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ON THE PROBLEM OF DETERMINING THE EXISTENCE OF IGNORABLE COORDINATES IN CONSERVATIVE DYNAMIC SYSTEMS

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The problem of using the form of the Lagrange function of a conservative dynamic system specified in an arbitrary generalized coordinate system as the basis for establishing the existence of a point coordinate transformation so as to have some of the new coordinates as ignorable is considered. Synge's conditions [1] are presented in a form similar to that of the invariant characteristic of the metric of rotation. A sequence of calculations is indicated by which it is possible to establish the existence ignorable coordinates in conservative systems with three degrees of freedom, and obtain in explicit form the related point transformation. Two applications of differential invariants and parameters for solving the problem with an arbitrary number of degrees of freedom are described.

Synge [1] established an effective criterion for solving this problem for a system with two degrees of freedom when the system force function is not constant (first results were published earlier [2-4[ ). The possibility of using Lie's group theory for solving this problem was suggested in [5], but now new results were obtained (see [3, 6]). Papers [7,8] should, also, be mentioned.

1. Let q^1 and q^2 be arbitrary generalized coordinates of a conservative dynamic system with two degrees of freedom. Let us, first, consider a system whose potential energy $V(q^1, q^2) \equiv \text{const}$.

As shown in [9],a set of surfaces S whose first basic quadratic form

$$ds^2 = 2Tdt^2 = a_{ij}dq^i dq^j \tag{1.1}$$

where T is the system kinetic energy and recurring subscripts indicate summation, exists in a three-dimensional Euclidean space.

Thus the problem of determining the existence of ignorable coordinates of metric (1.1) reduces to the problem of existence of an isometric image of surface S on some surface of revolution [10]. Such image exists when one of the following conditions is satisfied [10]:

$$K \equiv \text{const} \tag{1.2}$$

$$J(K, \Delta_1 K) = J(K, \Delta_2 K) = 0 \tag{1.3}$$

where

$$\Delta_1 f = a^{ij} \frac{\partial f}{\partial q^i} \frac{\partial f}{\partial q^j} \quad (\|a^{ij}\| = \|a_{ij}\|^{-1}) \tag{1.4}$$

$$\Delta_2 f = \frac{1}{c} \frac{\partial}{\partial q^i} \left(c a^{ij} \frac{\partial f}{\partial q^j} \right) \quad (c^2 = \det \|a_{ij}\|) \tag{1.5}$$

In these formulas $J(f, w)$ is the Jacobian of functions f and w, and K is the Gaussian curvature of surface S (curvature K is defined in terms of functions a_{ij} and its partial derivatives are determined by Gauss' formula [9,10]). Note that definitions (1.4) and (1.5) are valid for any n. When $K \not\equiv \text{const}$, the point transformation

$$Q^1 = K(q^1, q^2), \quad Q^2 = \chi(q^1, q^2)$$

where $\chi = \text{const}$ is the integral of the differential equation of the orthogonal trajectories of the line of Gaussian curvature $K = \text{const}$, reduces the system to a form containing the ignorable coordinate Q^2.

The definition of the ignorable coordinate implies that when $V \not\equiv \text{const}$ the fulfillment of conditions (1.2) or (1.3) is insufficient, although they are necessary. If $K \not\equiv \text{const}$, it is necessary, and with condition (1,3) it is also sufficient for $J(K, V) = 0$.

For the existence of a coordinate system with an ignorable coordinate it is generally necessary and sufficient that [1,3]

$$\begin{gathered} a^{ij}V_{1i}V_j / V_1 = a^{ij}V_{2i}V_j / V_2 \\ \lambda_{111}(V_2)^3 - (\lambda_{112} + \lambda_{121} + \lambda_{211})(V_2)^2 V_1 + \\ (\lambda_{221} + \lambda_{212} + \lambda_{122}) V_2 (V_1)^2 - \lambda_{222}(V_1)^3 = 0 \\ (\lambda_r = V_r / (a^{ij}V_iV_j), \; V_r = \partial V / \partial q^r) \end{gathered} \tag{1.6}$$

where V_{rs}, λ_{rs}, and λ_{rst} are covariant derivatives. The first of formulas (1.6) shows that equipotential lines are geodetically parallel, and the second is obtained from the condition that the curvature of each of such lines is constant.

Let us show that conditions (1.6) can be represented as

$$J(V, \Delta_1 V) = 0, \quad J(V, \Delta_2 V) = 0 \tag{1.7}$$

The differential parameters $\Delta_1 V$ and $\Delta_2 V$ are invariant to point transformations and, moreover, since in a coordinate system with an ignorable coordinate functions V, $\Delta_1 V$, and $\Delta_2 V$ depend on a single coordinate, formulas (1,7) (with the first relationships in (1.6) and (1.7) simply coincident) are valid for any selection of coordinates. Conversely, if conditions (1.7) are satisfied, the system is reducible to the form with an ignorable coordinate and, consequently, conditions (1.6) are valid.

To prove this we pass to new generalized coordinates

$$Q^1 = V(q^1, q^2), \quad Q^2 = \int \mu (a_{21}V_1 - a_{11}V_2)\, dq^1 + \mu (a_{22}V_1 - a_{12}V_2)\, dq^2 \tag{1.8}$$

(μ is the integrating coefficient). In new coordinates the Lagrange function is of the form

$$L = \frac{1}{2\Delta_1 V}\left[(Q^{\cdot 1})^2 + \frac{(Q^{\cdot 2})^2}{c^2\mu^2}\right] + Q^1 \tag{1.9}$$

We set in (1.8) $\mu = \omega(V) / c$. The condition of total differential with allowance for (1.4) and (1.5) yields

$$\omega \Delta_2 V + \frac{d\omega}{dV} \Delta_1 V = 0$$

hence

$$\mu = \frac{1}{c}\exp\left(-\int \frac{\Delta_2 V}{\Delta_1 V}\, dV\right)$$

and, consequently, Q^2 in formula (1.9) is an ignorable coordinate. The equivalence of conditions (1.6) and (1.7) is, thus, proved.

Hence for a system with two degrees of freedom the solution of the problem can be formulated in terms of differential parameters of functions K and V.

2. Let us consider a system with three degrees of freedom under condition that the potential energy $V \not\equiv \text{const}$. The sequence of calculations and transformations 1° — 5° specified below is used for solving the question of existence of a coordinate system whose at least one of the gereralized coordinates is ignorable. We note that non-fulfilment of the conditions 2° and 5° implies that none of coordinate systems contains ignorable coordinates.

1°. If in the Lagrange function of the system all coordinates q^1, q^2, and q^3 are positional, we pass to the new coordinates

$$Q^1 = Q^1(q^1, q^2, q^3), \quad Q^2 = Q^2(q^1, q^2, q^3), \quad Q = Q(V)$$

where Q is an arbitrary function and functions Q^1 and Q^2 satisfy the unique condition

$$\frac{\partial(Q^1, Q^2, V)}{\partial(q^1, q^2, q^3)} \neq 0$$

We then calculate the linear element of hypersurface $V = \text{const}$ of the system configuration space

$$ds^2 = E(Q^1, Q^2, Q)(dQ^1)^2 + 2F(\cdot)\, dQ^1 dQ^2 + G(\cdot)(dQ^2)^2 \tag{2.1}$$

and its Gaussian curvature $K = K(Q^1, Q^2, Q)$. We assume that $K|_{Q\text{-const}} = \text{const}$. In all calculations in transformations 2° and 3° coordinate Q is to be taken as a constant parameter.

2°. We determine $\Delta_1 K$ and $\Delta_2 K$ in the space with metric (2.1). It is necessary that

$$\Delta_1 K = \varphi(K, Q), \quad \Delta_2 K = \psi(K, Q)$$

3°. We determine function P by integrating the total differential

$$dP = \mu\left(F\frac{\partial K}{\partial Q^1} - E\frac{\partial K}{\partial Q^2}\right)dQ^1 + \mu\left(G\frac{\partial K}{\partial Q^1} - F\frac{\partial K}{\partial Q^2}\right)dQ^2$$

$$\mu = \frac{1}{\sqrt{EG - F^2}}\exp\left(-\int \frac{\psi(K, Q)}{\varphi(K, Q)}\, dK\right)$$

4°. We retain coordinate Q and pass from Q^1 and Q^2 to coordinates $N = N(K)$ (N is an arbitrary function) and P. The system Lagrange function assumes the form

$$L = {}^1\!/_2\,[G_{11}(N, Q)\, P^{\cdot 2} + G_{22}(N, Q)\, N^{\cdot 2} + 2G_{13}(P, N, Q)\, P^\cdot Q^\cdot + 2G_{23}(P, N, Q)\, N^\cdot Q^\cdot + G_{33}(P, N, Q)\, Q^{\cdot 2}] + w(Q) \tag{2.2}$$

5°. If coefficients G_{ij} satisfy the conditions

$$(G_{23})_P' = 0, \quad \left(G_{33} - \frac{G_{13}^2}{G_{11}}\right)_P' = 0 \qquad (2.3)$$

$$(G_{13})''_{P,P} = 0, \quad \left(\frac{G_{13}}{G_{11}}\right)''_{N,P} = 0$$

where the prime denotes a partial derivative of a function with respect to the related variable, then there exists the transformation

$$P = P\,(R, Q) \qquad (2.4)$$

which reduces Lagrangian (2.2) to the form with an ignorable coordinate R. Function (2.4) is determined by integrating total differentials.

We begin the substantiation of sequence 1° — 5° by proving the following lemma.

Lemma. When a conservative dynamic system has an ignorable coordinate and the Gaussian curvature of an arbitrary equipotential hypersurface $V = \text{const}$ is not a constant quantity for that hypersurface, there exists a coordinate system in which the Lagrangian

$$L = {}^1/_2\,[B_{11}(N, Q)\,R^{\cdot 2} + B_{22}(\cdot)\,N^{\cdot 2} + 2B_{13}(\cdot)\,R^{\cdot}Q^{\cdot} + 2B_{23}(\cdot)\,N^{\cdot}Q^{\cdot} + B_{33}(\cdot)\,Q^{\cdot 2}] + w(Q) \qquad (2.5)$$

Proof. Let us assume that among the system coordinates q^1, q^2, q^3 the coordinate q^1 is ignorable. Since $V \not\equiv \text{const}$, we can assume $\partial V/\partial q^3 \neq 0$. We carry out the substitution

$$q^1 = q^1,\ q^2 = q^2,\ Q = Q\,(V)$$

The system Lagrangian now becomes

$$L = {}^1/_2\,[b_{11}\,(q^2, Q)\,(q^{\cdot 1})^2 + 2\,b_{12}\,(\cdot)\,q^{\cdot 1}q^{\cdot 2} + b_{22}\,(\cdot)\,(q^{\cdot 2})^2 + 2\,b_{13}\,(\cdot) \times q^{\cdot 1}Q^{\cdot} + 2\,b_{23}\,(\cdot)\,q^{\cdot 2}Q^{\cdot} + b_{33}\,(\cdot)\,Q^{\cdot 2}] + w\,(Q) \qquad (2.6)$$

The linear element of hypersurface $V = \text{const}$

$$ds^2 = b_{11}\,(dq^1)^2 + 2\,b_{12}dq^1dq^2 + b_{22}\,(dq^2)^2 \qquad (2.7)$$

does not explicitly contain the coordinate q^1. Hence the Gaussian curvature $K = K(q^2, Q)$, and in the space with metric (2.7) by virtue of conditions (1.3)

$$\Delta_1 K = \varphi\,(K, Q), \quad \Delta_2 K = \psi\,(K, Q)$$

The differential equation of the orthogonal trajectories of lines of constant Gaussian curvature on the hypersurface $V = \text{const}$

$$b_{11}dq^1 + b_{12}dq^2 = 0$$

has, obviously, the integral $R = \text{const}$, where

$$R = q^1 + A\,(q^2, Q)$$

In coordinates R, $N = N\,(K)$, Q Lagrangian (2.6) assumes the form (2.5).

The proof of the lemma shows that, when a conservative dynamic system has an ignorable coordinate, then in the system space configuration a linear element of the equipotential hypersurface is also reducible to the form with an ignorable coordinate. Hence in accordance with (1.3) conditions 2° are necessary. It is, moreover evident that any two general solutions P and R of the differential equation of orthogonal trajectories of lines $K = \text{const}$ on hypersurface $V = \text{const}$ are related by formula (2.4). Comparing formulas (2.2) and (2.5), for the system Lagrangian we obtain

$$G_{22} = B_{22},\ G_{23} = B_{23} \tag{2.8}$$

$$G_{11}\left(\frac{\partial P}{\partial R}\right)^2 = B_{11},\quad G_{11}\frac{\partial P}{\partial R}\frac{\partial P}{\partial Q} + G_{13}\frac{\partial P}{\partial R} = B_{13} \tag{2.9}$$

$$G_{11}\left(\frac{\partial P}{\partial Q}\right)^2 + 2G_{13}\frac{\partial P}{\partial Q} + G_{33} = B_{33}$$

In these formulas only G_{ij} represent known functions. Let us show that they must satisfy conditions (2.3).

The first of conditions (2.3) follows from (2.8), and from the first two of formulas (2.9) and (2.4) we obtain

$$\frac{\partial P}{\partial R} = \frac{f}{\sqrt{G_{11}}},\quad \frac{\partial P}{\partial Q} = -\frac{G_{13}}{G_{11}} + \frac{g}{\sqrt{G_{11}}},\quad \frac{\partial P}{\partial N} = 0 \tag{2.10}$$

$$(f = \sqrt{B_{11}},\ g = B_{13}/\sqrt{B_{11}})$$

Substitution of the expression $\partial P / \partial Q$ into the third of formulas (2.9) yields

$$G_{33} - \frac{G_{13}^2}{G_{11}} = B_{33} - g^2$$

from which follows the second of conditions (2.3). The condition of equality of related mixed derivatives of function P yield

$$\left(\frac{1}{\sqrt{G_{11}}}\right)_N' f + \frac{f_N'}{\sqrt{G_{11}}} = 0,\quad \left(\frac{1}{\sqrt{G_{11}}}\right)_Q' f + \frac{f_Q'}{\sqrt{G_{11}}} = \tag{2.11}$$

$$-\left(\frac{G_{13}}{G_{11}}\right)_P' \frac{f}{\sqrt{G_{11}}},\quad -\left(\frac{G_{13}}{G_{11}}\right)_N' + \left(\frac{1}{\sqrt{G_{11}}}\right)_N' g + \frac{g_N'}{\sqrt{G_{11}}} = 0$$

and furthermore

$$f_P' = 0,\quad g_P' = 0 \tag{2.12}$$

The problem is thus reduced to the analysis of compatibility of Eqs. (2.11) and (2.12) with unknown functions f and g.

Using the equality of related second order mixed derivatives of function $\ln f$, from Eqs. (2.11) and (2.12) we obtain three relationships, one of which yields the third of conditions (2.3), the second is the corollary of the last of Eqs. (2.11), and the third is identically satisfied. The condition of compatibility of Eqs. (2.11) and (2.12) relative to function g yields the fourth of conditions (2.3). The necessity of condition (2.3) is proved.

Conversely, if coefficients G_{ij} satisfy conditions (2.3), there exists transformation (2.4) which reduces Lagrangian (2.2) to the form with an ignorable coordinate R. Function (2.4) can be determined by integrating the compatible system of Eqs. (2.10) in which the functions f and g are obtained by the integration of Eqs. (2.11) and (2.12), and the system of these equations is also compatible.

3. The problem of determining the existence of ignorable coordinates in systems with an arbitrary number n of degrees of freedom can be solved if the differential invariants and potential energy parameters of the system are known.

Below we consider cases when n and $n-1$ invariants and parameters are known.

Certain definitions are required for the subsequent analysis [10, 11]. Let $q^1, \ldots, q^n$ be the generalized coordinates of a dynamic system and $a_{ij}(q)$ be the coefficients of its doubled kinetic energy.

Definition 1. The term differential invariant denotes any expression of φ consisting of coefficients a_{ij} and their partial derivatives with respect to variables q^r up to some order and which, as the result of the arbitrary substitution

$$Q^j = Q^j(q^1, \ldots, q^n) \quad (j = 1, \ldots, n) \tag{3.1}$$

is transformed into the same expression consisting of coefficients A_{kl} of converted form and of its partial derivatives with respect to the new variables Q^s, i.e.

$$\varphi\left(\ldots, A_{ij}, \ldots, \frac{\partial A_{ij}}{\partial Q^r}, \ldots, \frac{\partial^2 A_{ij}}{\partial Q^r \partial Q^s}, \ldots\right) = \varphi\left(\ldots, a_{ij}, \ldots, \frac{\partial a_{ij}}{\partial q^r}, \ldots, \frac{\partial^2 a_{ij}}{\partial q^r \partial q^s}, \ldots\right)$$

where the dots denote other terms of the indicated kind.

Definition 2. When the expression of φ contains in addition arbitrary functions $f, w, \ldots$ of variables q^r and their partial derivatives with respect to such variables up to some order, and which by the use of substitution (3.1) is converted into the same expression consisting of coefficients of converted form and of partial derivatives of the same functions with respect to new variables, that expression is called the differential parameter of functions $f, w, \ldots$. This definition is represented by the equality

$$\varphi\left(\ldots, A_{ij}, \ldots, \frac{\partial A_{ij}}{\partial Q^r}, \ldots, \frac{\partial^2 A_{ij}}{\partial Q^r \partial Q^s}, \ldots, F, W, \ldots \right.$$
$$\left. \ldots, \frac{\partial F}{\partial Q^k}, \ldots, \frac{\partial W}{\partial Q^t}, \ldots\right) =$$
$$\varphi\left(\ldots, a_{ij}, \ldots, \frac{\partial a_{ij}}{\partial q^r}, \ldots, \frac{\partial^2 a_{ij}}{\partial q^r \partial q^s}, \ldots, f, w, \ldots \right.$$
$$\left. \ldots, \frac{\partial f}{\partial q^k}, \ldots, \frac{\partial w}{\partial q^t}, \ldots\right)$$

For example, the scalar curvature of the Riemann space of the system configuration is a differential invariant [9]. Formulas (1.4) and (1.5) are examples of differential parameters of arbitrary function f. The expression

$$\nabla(f, w) = a^{ij} \frac{\partial f}{\partial q^i} \frac{\partial w}{\partial q^j}$$

is called mixed differential parameter of the first order of functions f and w. Using all possible combinations of operators Δ_1, Δ_2, and Δ to the already known differential invariants and parameters it is possible to obtain differential invariants and parameters of higher orders.

The above definition with the definition of the ignorable coordinate of a dynamic system yield the following theorem. If a dynamic system with n degrees of freedom has n functionally independent differential invariants and parameters of potential energy V (including also function V) $I_1, I_2, \ldots, I_n$, then none of the coordinate systems contains ignorable coordinates.

Thus a dynamic system can have ignorable coordinates only when the number of independent functions $I_1, \ldots, I_k$ in it is smaller than the number of generalized coordinates. We shall show that when $k = n - 1$ the existence of an ignorable coordinate is effectively determined by differentation and simple algebraic operations. For this we use the Whittaker theorem [4]: if a dynamic system admits the integral

$$c_i\,(q^1, \ldots, q^n)\, q^{\cdot i} = \text{const} \tag{3.2}$$

there exists transformation (3.1) by which the system Lagrangian is transformed to the form with the ignorable coordinate Q^n. Note that the theorem is invalid [12] when (3.2) is a partial integral of Lagrange equations of the system. Structure of the Lagrangian was investigated for this case in [2-14] and the obtained results are given in the survey [15].

Let us now assume that $n - 1$ independent invariant functions I_r are known. Their expressions in terms of new coordinates do not contain Q^n, i.e.

$$\frac{\partial I_r}{\partial q^j}\frac{\partial q^j}{\partial Q^n} = 0 \quad (r = 1, \ldots, n-1)$$

hence

$$\partial q^j/\partial Q^n = m^j\omega \quad (j = 1, \ldots, n)$$

where $m^j\,(q^1, \ldots, q^n)$ are known functions and $\omega\,(q^1, \ldots, q^n)$ is some unknown function which is not identically zero. In new coordinates integral (3.2) is of the form

$$\frac{\partial L}{\partial Q^{\cdot n}} = \text{const}\left(\frac{\partial L}{\partial q^{\cdot j}}\frac{\partial q^{\cdot j}}{\partial Q^{\cdot n}} = \text{const}\right)$$

where L is the system Lagrange function. But for any j

$$\frac{\partial q^{\cdot j}}{\partial Q^{\cdot n}} = \frac{\partial q^j}{\partial Q^n} = m^j\omega$$

hence integral (3.2) can be represented as

$$\frac{\partial L}{\partial q^{\cdot j}}\, m^j\omega = \text{const} \tag{3.3}$$

To ascertain the existence of an ignorable coordinate it is necessary, according to Whittaker's theorem, to check whether the Lagrange equations admit integral (3.3). Differentiating (3.3) with respect to time on the basis of Lagrange equations we obtain

$$\frac{\partial L}{\partial q^j}\, m^j\omega + \frac{\partial L}{\partial q^{\cdot j}}\,\frac{\partial\,(m^j\omega)}{\partial q^l}\, q^{\cdot l} = 0$$

This relationship must be an identity. Equating in it coefficients at generalized velocities to zero, we obtain not more than $n\,(n+1)/2$ equations in partial derivatives of the first order with respect to the unknown function ω. Analysis of the compatibility of equations is effected with the use of Jacobi's brackets [16]. If the system of equations proves to be compatible only for $\omega=0$, the cônsidered dynamic system does not contain linear integrals. When, however, the system of equations is compatible for $\omega = \omega_0 \not\equiv 0$, expression (3.3) is an integral of the dynamic system when $\omega = \omega_0$.

Example. Let us consider the Lagrange function

$$L = {}^1/_2 (x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) - {}^1/_2 (k_1 x^2 + k_2 y^2 + k_3 z^2)$$

where k_1, k_2, and k_3 are constants. Such function defines, for instance, the small oscillations of a particle suspended on three mutually perpendicular springs of k_1, k_2, and k_3 stiffness.

We introduce the invariant functions

$$V = {}^1/_2 (k_1 x^2 + k_2 y^2 + k_3 z^2), \quad \Delta_1 V = k_1{}^2 x^2 + k_2{}^2 y^2 + k_3{}^2 z^2$$
$$\nabla (V, \Delta_1 V) = 2 (k_1{}^3 x^2 + k_2{}^3 y^2 + k_3{}^3 z^2)$$

When the three numbers k_1, k_2, and k_3 are different, these functions are independent and, consequently, the system cannot have an ignorable coordinate.

If $k_1 = k_2 \neq k_3$, functions V and $\Delta_1 V$ are independent. Proceeding in this manner we obtain

$$\frac{\partial x}{\partial Q^3} = y\omega, \quad \frac{\partial y}{\partial Q^3} = -x\omega, \quad \frac{\partial z}{\partial Q^3} = 0$$

and formulas (3.3) in the form

$$\omega (yx^{\cdot} - xy^{\cdot}) = \text{const} \tag{3.4}$$

When $\omega \not\equiv 0$, the conditions of existence of integral (3.4)

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial y} = \frac{\partial \omega}{\partial z} = 0$$

are compatible and, consequently, the system of coordinates with an ignorable coordinate exists. In cylindrical coordinates r, φ, z the Lagrange function obviously does not contain φ.

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